

FRAME AMBIGUITY IN OPEN GROMOV-WITTEN INVARIANTS

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ABSTRACT. We consider Open Gromov-Witten invariants for noncompact Calabi-Yau in the case the Lagrangian has the topology of $\mathbb{R}^2 \times S^1$. The definition of the invariant involves the choice of a frame for the Lagrangian, in accord with string theory.

Our result applies to the examples arising from Large N -duality. In particular it leads to knot and link invariants counting holomorphic curves.

1. INTRODUCTION

For X a *compact* Calabi-Yau three-fold and L a Maslov index zero Lagrangian submanifold, in [7], [8], we construct the Open Gromov-Witten invariants associated to the pair (X, L) . These invariants are the natural mathematical formulation of the Open Topological String partition function defined by Witten [16]. Our construction formalizes the result of Witten expressing the open topological amplitudes in terms of Wilson loop integrals of the Chern-Simons theory living on L .

In [1] and [2], Aganagic, Klemm and Vafa consider examples of *noncompact* pairs (X, L) in the case that L has the topology of $\mathbb{R}^2 \times S^1$. In these examples it is realized that the Open Topological partition function is not uniquely determined, but it is affected by an ambiguity. The appearance of this ambiguity is tied with the usual fact that a problem on a noncompact space is well defined only after some boundary conditions at infinity are fixed.

The computations of [1] and [2] were made in two different ways, using mirror symmetry and using large N duality. In mirror symmetry the ambiguity is viewed as a choice of boundary condition at infinity on the mirror non-compact brane. In large N duality the ambiguity manifests in the choice of the frame of the knot associated to the Lagrangian submanifold.

In this paper we give a mathematical definition of Open Gromov-Witten invariants for non-compact Lagrangians L whose connected components have the topology of $\mathbb{R}^2 \times S^1$. We extend the construction of [7] to this case. Here we need to make the assumption that Gromov compactness holds. We provide an explanation of the frame ambiguity entirely in symplectic topology, without using any (unproved) duality. Our result provides the natural definition of Open Gromov-Witten invariants used in Large N duality ([13], [11], [9]). In particular it leads to knot and link invariants counting holomorphic curves. In the past there have been other attempts to define such invariants in symplectic topology, however they lead to knot invariants which seem distinct from the original invariant (see [12]).

We consider first the problem of defining the linking number of two curves on L . The linking number is defined after a choice of compactification of L to a 3-sphere.

We introduce the topological notion of frame of L . To each frame corresponds a compactification of L and therefore a definition of linking number of curves. We apply these considerations to define Open Gromov-Witten invariants for the pair (X, L) as generalized linking number of a system of chains following the same recipe of [7]. An analogous construction can be applied every time that $H_2(L, \mathbb{Q}) = 0$.

We can also understand the appearance of the frame ambiguity using Chern-Simons theory, more in relation with [16] (see [8]). To define the open Gromov-Witten invariants for the pair (X, L) we need in particular to define the Chern-Simons propagator living on L . As usual in not compact situation, the problem involves the choice of boundary conditions at infinity. A way to proceed consist in define the propagator on L restricting the propagator defined on a compactification of L .

2. FRAMES AND LINKING NUMBERS

Let L be a three manifold with the topology of $\mathbb{R}^2 \times S^1$. In this section we consider the problem of define the linking number of two curves on L . We will show that (essentially because $H_2(L) = 0$) it is possible to extend the usual definition of linking number on S^3 to this topology but it can not be done in a unique way.

Let $L' \subset L$ be a complement of compact set of L such that

$$L' \cong \mathbb{R} \times S^1 \times S^1.$$

In particular $H_1(L') \cong \mathbb{Z} \times \mathbb{Z}$. Fix a generator v of the kernel of $H_1(L') \rightarrow H_1(L)$:

$$\text{Span}_{\mathbb{Z}}\{v\} = \ker\{H_1(L') \rightarrow H_1(L)\}.$$

Definition 1. A frame of L is an element $f \in H_1(L')$ such that

$$(1) \quad \text{Span}_{\mathbb{Z}}\{v, f\} = H_1(L').$$

We identify two frames if they differ by a sign.

Fix an isomorphism $H_1(L') \cong \mathbb{Z} \times \mathbb{Z}$ and write $f = (f_1, f_2)$ and $v = (v_1, v_2)$. Condition (1) is equivalent to

$$(2) \quad f_1 v_2 - f_2 v_1 = \pm 1.$$

We now consider the problem of define linking numbers of curves on L after we pick a frame of L . By a linking number of curves on L we mean a law that associates to every pair of embedded disjoint curves on L an integer that jumps by ± 1 when two curves cross, with sign determinate according to the sign of the cross.

Lemma 2. To each frame of L is associated a definition of linking number of curves on L .

Proof. The linking number between two curves on S^3 is defined by shrinking the curves and counting intersection numbers.

Not all the curves in L can be shrunked. However we can adapt the definition of S^3 declaring zero curves at infinity representing some multiple of f in $H_1(L')$. More precisely we can proceed as follows.

Two curves γ_1^0 and γ_2^0 on L' representing a multiple of the homological class f are defined unlinked if for each compact set K of L there exists an homotopy that does not intersect γ_2^0 , of γ_1^0 to a curve on $L \setminus K$, and there exists a homotopy of γ_2^0 that does not intersect γ_1^0 , to a curve in $L \setminus K$.

Observe that there are plenty of pairs of curves with this property. For example this holds for each pair (γ_1^0, γ_2^0) with γ_1^0 in $\{t_1\} \times S^1 \times S^1$ and γ_2^0 in $\{t_2\} \times S^1 \times S^1$ for $t_1 \neq t_2$ and representing a multiple of the homological class f (here we fixed an isomorphism $L' \cong \mathbb{R} \times S^1 \times S^1$).

Fix a pair (γ_1^0, γ_2^0) of curves on L' representing the homological class f and unlinked in the sense above. Given two curves γ_1 and γ_2 on L , there exist $c_1, c_2 \in \mathbb{Z}$ such that $[\gamma_1] = c_1 f$ and $[\gamma_2] = c_2 f$ in $H_1(L)$. For $i = 1, 2$, γ_i is homotopic to $c_i \gamma_i^0$, that is there exist a singular chain $B_i \in C_2([0, 1] \times L)$ such that

$$\partial B_i = c_i \{1\} \times \gamma_i^0 - \{0\} \times \gamma_i.$$

Define the linking number of γ_1, γ_2 as the intersection number of B_1 and B_2

$$\text{link}(\gamma_1, \gamma_2) = (B_1 \times_{[0,1]} B_2) \cap \Delta = B_1 \cap B_2.$$

From the definition it follows that $\text{link}(\gamma_1, \gamma_2)$ does not depend on the choice of the pair (γ_1^0, γ_2^0) . In fact if (γ_1^0, γ_2^0) and (γ_1^1, γ_2^1) are two pairs of unlinked curves in the sense above, there exists a homotopy between γ_1^0 and γ_1^1 and a homotopy between γ_2^0 and γ_2^1 not intersecting each other. \square

We can also see the correspondence between frames and linking numbers considering compactifications of L to spheres. Observe that, if we have a compactification of L to an S^3

$$(3) \quad \bar{f} : L \hookrightarrow S^3.$$

we can define the linking number of curves on L considering them as curves on S^3 . Of course the result will depend on the compactification.

The connection with the previous definition is the following. Frames of L induces compactifications of L

$$\text{frames of } L \rightsquigarrow \text{compactifications of } L.$$

To a frame f it corresponds a compactification (3) such that $S^3 \setminus L \cong S^1$ and

$$(4) \quad \text{Span}_{\mathbb{Z}}\{f\} = \ker(H_1(L') \rightarrow H_1(S^3 \setminus K)).$$

Fix a diffeomorphism $L \cong \mathbb{R}^2 \times S^1$. The compactification S^3 is made gluing to L another copy of $\mathbb{R}^2 \times S^1$ over $\mathbb{R}^2 \setminus \{(0, 0)\}$ in the following way. Observe that $(\mathbb{R}^2 \setminus \{(0, 0)\}) \times S^1 \cong \mathbb{R} \times T^2$, where T^2 is the torus of dimension two. The elements $f, v \in H_1(L')$ define a basis (over \mathbb{Z}) of $H_1(T^2)$. Let $A_f : T^2 \rightarrow T^2$ be a map that interchange f and v up to sign and preserves the orientation. Define

$$S^3 = L \sqcup_g (\mathbb{R}^2 \times S^1)$$

where g is the diffeomorphism $g : (\mathbb{R}^2 \setminus \{(0, 0)\}) \times S^1 \rightarrow (\mathbb{R}^2 \setminus \{(0, 0)\}) \times S^1$ given by

$$(5) \quad (r, (s, t)) \rightarrow (-r, A_f(s, t))$$

where $r \in \mathbb{R}$ and $s, t \in S^1$. It is easy to see that the definition of linking number of Lemma (2) agrees with the one coming from (3).

3. OPEN GROMOV-WITTEN INVARIANTS

Let X be a noncompact Calabi-Yau three-folds and L a Lagrangian submanifold with topology of $\mathbb{R}^2 \times S^1$ and Maslov index zero. In this section we want to define the Open Gromov Witten invariants of the pair (X, L) following the construction of [7].

Since the space is noncompact we need to make the technical assumption that Gromov compactness holds for closed holomorphic curves as well as for holomorphic curves with boundary mapped on L . The problem of Gromov compactness for noncompact manifolds is considered for example in [14].

As in [7], in order to solve the problem of the bubbling of spheres from constant disks, we assume that L is homological trivial in X . Fix a $B \in C_4(X)$ such that

$$(6) \quad \partial B = L.$$

The necessary modifications are the same of [7] and they will not be reported here.

3.1. Multi-curves. The set of decorated graphs \mathcal{G} is defined as in [7]. For each $G \in \mathcal{G}$, define the moduli space of multi-curves $\overline{\mathcal{M}}_G$ as in [7]

$$\overline{\mathcal{M}}_G = \left(\prod_{i \in I} \overline{\mathcal{M}}_i \right) / \text{Aut}(G).$$

In order to be able to count the winding number of the boundary components of the multi-curves we need to refine the definition of evaluation map on the boundary marked points. For this we need to encode the homotopy class of the boundary paths between consecutive boundary marked points as follows.

Let $\{x_e\}_{e \in H(G)} \in L^{H(G)}$. For $v \in V(G)$ let $H(v) = \{e_1, \dots, e_{|H(v)|}\}$ be the cyclic order of the half edges starting in v . Define

$$\pi_1(v) = \prod_i \pi_1(L, x_{e_i}, x_{e_{i+1}})$$

where, for each i ,

$$\pi_1(L, x_{e_i}, x_{e_{i+1}}) = \{\text{homotopy classes of paths between } x_{e_i} \text{ and } x_{e_{i+1}}\}.$$

In the particular case that $H(v)$ is empty we set $\pi_1(v) = \pi_1(L)$.

Let $\tilde{L}^{H(G)}$ be the covering space of $L^{H(G)}$

$$(7) \quad \pi : \tilde{L}^{H(G)} \rightarrow L^{H(G)}$$

with fibers

$$(8) \quad \pi^{-1}(\{x_e\}_{e \in H(G)}) = \prod_{v \in V(G)} \pi_1(v).$$

Define

$$\tilde{L}_G = \tilde{L}^{H(G)} / \text{Aut}(G).$$

The evaluation map on the boundary punctures can be refined considering the boundary paths between consecutive punctures. This leads to the natural map

$$(9) \quad \text{ev} : \overline{\mathcal{M}}_G \rightarrow \tilde{L}_G.$$

3.2. Systems of singular chains. For e internal edge of G , there exists a natural lifting $\tilde{\Delta}_e$ of the big diagonal Δ_e of L_G to a smooth simplicial chain of the orbifold \tilde{L}_G .

Consider e connecting two vertices v_0 and v_1 (it can be $v_0 = v_1$). Let e_0, e_1 the puncture associated to e , in the boundary components associated to v_0, v_1 . Let e_i^+, e_i^- be the half edges after and before e_i respectively. On $\tilde{\Delta}_e$, since $x_{e_0} = x_{e_1}$, we can define the gluing maps

$$\begin{aligned} \pi_1(L, x_{e_0}^-, x_{e_0}) \times \pi_1(L, x_{e_1}, x_{e_1}^+) &\rightarrow \pi_1(L, x_{e_0}^-, x_{e_1}^+) \\ \pi_1(L, x_{e_1}^-, x_{e_1}) \times \pi_1(L, x_{e_0}, x_{e_0}^+) &\rightarrow \pi_1(L, x_{e_1}^-, x_{e_0}^+). \end{aligned}$$

The cases where at least one of v_0 and v_1 have valence equal to one are specials. If for example v_0 has valence one, the gluing map is

$$\pi_1(L, x_{e_1}^-, x_{e_1}) \times \pi_1(L, x_{e_0}, x_{e_0}) \times \pi_1(L, x_{e_1}, x_{e_1}^+) \rightarrow \pi_1(L, x_{e_1}^-, x_{e_1}^+).$$

The gluing maps above lead to the map

$$(10) \quad \tilde{\Delta}_e \rightarrow \tilde{L}_{G/e}.$$

We use this map to extend the definition of system of chains.

Definition 3. *A system of singular chains*

$$W_G = \{W_G\}_{G \in \mathcal{G}}$$

is a collection of singular chains $W_G \in C_{|E(G)|}(\tilde{L}_G, \mathfrak{o}_G)$ with twisted coefficients in \mathfrak{o}_G . Fulfilling the following properties:

- (a) For each $G \in \mathcal{G}$, and e an internal edge of G , W_G intersects $\tilde{\Delta}_e$ transversely. Let $\partial_e W_G$ be the singular chain on $\tilde{L}_{G/e}$ defined by

$$W_G \cap \tilde{\Delta}_e \in C_*(\tilde{L}_{G/e}, \mathfrak{o}_{G/e})$$

using the map (10).

- (b) The following identity holds as currents

$$(11) \quad \partial W_G = \sum_{G'/e'=G} \partial_{e'} W_{G'}.$$

Here the sum is taken over all the pairs (G', e') with $e' \in E(G')$ such that $G'/e' \cong G$.

In the same way we can extend the definition of homotopies of system of chains and equivalences of homotopies.

Exactly as in [7] we can constrain the perturbation of the Kuranishi spaces $\{\overline{\mathcal{M}}_G\}_{G \in \mathcal{G}}$. Using these perturbations and the map (9) we define a system of chains

$$(12) \quad W_G = \{W_G\}_{G \in \mathcal{G}}$$

with $W_G \in C_*(\tilde{L}_G, \mathfrak{o}_G)$.

As in [7] we have

Proposition 4. (12) defines a system of chains. Different choices of perturbations lead to homotopic systems of chains, with homotopy uniquely determined up to equivalence.

Now we can define the Open Gromov-Witten invariants.

Proposition 5. *Let f be a frame of L . The rational numbers $F_{g,n_1,\dots,n_h}^f(A)$ are defined counting multi-curves in the relative homology class $A \in H_2(X, L)$ of genus g , h boundary components and with the i -th boundary component winding n_i times around the nontrivial cycle of L . These numbers depend only on the pair (X, L) and f (and the choice of B in (6)).*

Proof. As in [7] the rational numbers $F_{g,n_1,\dots,n_h}^f(A)$ are defined in terms of generalized linking numbers for the system of chains (12).

Recall that to define the generalized linking number of a system of chain in [7] we recursively contract the chains starting with the component associated to the graph with a maximal number of edges.

In our case the singular chains are not homologically trivial. However we can adapt the procedure of Lemma (2) to get an invariant depending of the choice of the frame f .

The models of unlinked system of chains depends on the choice of some data. For each decorated graph G choice real numbers

$$(13) \quad \{t_e\}_{e \in H(G)}.$$

Given a vertex v_0 of G associated to a disk of area zero, and a real number s , define a new (13) by changing t_e to $t_e + s$ for each $e \in H(v_0)$. This generate an equivalence relation on the set of data (13).

We require that the data (13) satisfies the following compatibility condition for each $e_0 \in H(G)$ oriented edge ending to a vertex v_0 associated to the boundary of a disk of area zero. Let e_1 be e_0 with opposite orientation. For $e \in H(G)$ with $e \neq e_0$, so that e defines an edge \tilde{e} of G/e_0 . If e start from v_0 , we require $t_{\tilde{e}} = t_e - t_{e_1} + t_{e_0}$. Otherwise we require $t_{\tilde{e}} = t_e$.

Using a generic choice of data (13) with the above compatibility condition we now construct a system of chains. We first define a singular chain $W_v \in C_*(L^{H(v)})$ for each vertex v of a graph G .

Let e_1, e_2, \dots, e_k be a cyclic order of the edges starting from v . Suppose first that v corresponds to the boundary component of a curve of positive area. As in Lemma (2) let $L' \subset L$ be a complement of a compact subset of L such that $L_0 \cong \mathbb{R} \times S_1 \times S_1$. Fix a curve $\gamma_0 : S_1 \rightarrow S_1 \times S_1$ representing the frame f . Let W_v be represented by the geometric cycle

$$(s_{e_1}, s_{e_2}, \dots, s_{e_k}) \rightarrow ((t_{e_1}, \gamma_0(s_{e_1})), (t_{e_2}, \gamma_0(s_{e_2})), \dots, (t_{e_k}, \gamma_0(s_{e_k})))$$

Assume now that v is associated to the boundary of a disk of area zero. The geometric cycle

$$\begin{aligned} \mathbb{R}^+ \times S_1 \times S_1 &\rightarrow (\mathbb{R} \times S_1 \times S_1)^k \subset L^k \\ (t, p) &\rightarrow (t, p), (t_{e_1} - t_{e_2} + t, p), \dots, (t_{e_k} - t_{e_{k-1}} + t, p), (t_{e_1} - t_{e_k} + t, p) \end{aligned}$$

can be extended to L . Let W_v be the corresponding singular chain.

Define

$$W_G^{unlinked} = \prod_{v \in V(G)} W_v.$$

Using an iterative argument as in [7] it is not hard to show that any system of chain is homotopic to a unique system of chain W_G^0 such that, for each G , W_G^0 is some rational number times $W_G^{unlinked}$

$$W_G^0 \sim W_G^{unlinked}.$$

Different choices of (13) lead to different W_G^0 . However the 0-chains associated to graphs without edges do not change.

The construction above applied to the system of chain (12) leads therefore to invariants of the triple (X, L, f) . We define Open Gromov-Witten invariants $F_{g,n_1,\dots,n_h}^f(A)$ by the formula

$$W_{G_{g,h}}^0 = \sum F_{g,n_1,\dots,n_h}^f(A)[n_1, \dots, n_h]$$

where for each $(g; h)$ let $G_{g,h}$ be the unique decorated graph with no edges of genus g and h boundary components and $[n_1, \dots, n_h]$ is the singular 0-chain with support in the point $(n_1, \dots, n_h) \in \tilde{L}_{G_{g,h}}$ (observe that $\tilde{L}_{G_{g,h}} = \pi_1(L)^h \cong \mathbb{Z}^h$).

□

It is straightforward to extend the preview result in the case that L is a disjoint union of submanifolds

$$L = \sqcup_{\alpha=1}^k L_\alpha$$

where each component L_i has the topology of $S^1 \times \mathbb{R}^2$. In this case we assume that each component L_i is homological trivial. For each α let f_α be a frame for L_α . As before we can define the invariants

$$F_{g,\vec{n}^{(1)},\dots,\vec{n}^{(k)}}^{f_1,\dots,f_k}(A).$$

Under the weaker assumption that L is homological trivial but not all its single components L_α , it is not possible to distinguish on which connected component of L the boundary components with zero winding number are mapped.

4. KNOTS AND LINKS INVARIANTS

Gopakumar and Vafa in [3] and [4] conjectured that the closed topological strings amplitudes (closed Gromov-Witten invariants) of the resolved conifold

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

can be computed in terms of the Chern-Simons invariants of S^3 (here $\mathcal{O}(-1)$ is the tautological bundle over \mathbb{CP}^1). This duality is referred to as Large N duality. The conjecture was later extended to Open amplitudes in [13], [11], [9]. The result in particular predicts the existence of knot and link invariants counting holomorphic curves. In this section we apply the result of the previous section to give a mathematical definition of these invariants.

Let \mathcal{K} be a knot in S^3 represented by the curve

$$\gamma : S^1 \rightarrow S^3.$$

Large N duality associate to \mathcal{K} a Lagrangian submanifold of the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

$$\mathcal{K} \rightsquigarrow L_{\mathcal{K}}$$

such that the topological open string invariants of $L_{\mathcal{K}}$ can be computed in terms of Chern-Simons invariants of \mathcal{K} . The Lagrangian $L_{\mathcal{K}}$ was constructed for the unknot in [13] and generalized to all knots in [15] and [6]. For our purpose it is important to notice that as a manifold $L_{\mathcal{K}}$ is the same of the conormal Lagrangian $\mathcal{C}_{\mathcal{K}} \subset T^*S^3$ of \mathcal{K} (see [6]).

Lemma 6. *Each frame of the knot \mathcal{K} defines a frame of the Lagrangian $L_{\mathcal{K}}$. This correspondence is one to one*

$$(14) \quad \text{frames of } \mathcal{K} \leftrightarrow \text{frames of } L_{\mathcal{K}}$$

Proof. Let

$$f : S^1 \rightarrow \gamma^*(TS^3)/\langle \dot{\gamma} \rangle$$

be a frame of \mathcal{K} .

Since $L_{\mathcal{K}}$ and $\mathcal{C}_{\mathcal{K}}$ are the same as manifolds, we need to associate to f a frame of $\mathcal{C}_{\mathcal{K}}$. Define

$$\tilde{\mathcal{C}}_{\mathcal{K}} = \mathcal{C}_{\mathcal{K}} \setminus K$$

where K is compact with the topology of $S^1 \times Q$ with Q contractible (as in Definition 1). We need to associate to f an element of $H_1(\tilde{\mathcal{C}}_{\mathcal{K}})$ such that condition (2) holds.

Let η be a section

$$\eta : S^1 \rightarrow \gamma^*(T^*S^3)$$

such that for each $s \in S^1$

$$\langle \eta(s), \dot{\gamma}(s) \rangle = 0$$

$$\langle \eta(s), f(s) \rangle = 0.$$

The graph of η defines a one cycle of $\tilde{\mathcal{C}}_{\mathcal{K}}$ that satisfies condition (2), and therefore a frame of $\mathcal{C}_{\mathcal{K}}$. □

In [6] it is proved that Gromov compactness holds for the pair $(X, L_{\mathcal{K}})$ for each knot \mathcal{K} . Moreover since $H_3(X) = 0$ the Lagrangians $L_{\mathcal{K}}$ is homological trivial.

Proposition 7. *Let \mathcal{K}^f be a framed knot in S^3 . The Open Gromov-Witten invariants $F_{g, (l_1, \dots, l_h)}^f$ of genus g and winding numbers l_1, \dots, l_h are defined.*

Observe that since $H_4(X) = 0$, the invariants do not depend on the choice of B in (6).

It is straightforward extend the result to the case of link invariants.

Proposition 8. *Let $\mathcal{K}_1^{f_1}, \dots, \mathcal{K}_k^{f_k}$ be a framed link in S^3 . The Open Gromov-Witten invariants $F_{g, (\vec{n}^{(1)}, \dots, \vec{n}^{(k)})}^{f_1, \dots, f_k}$ of genus g and winding numbers $(\vec{n}^{(1)}, \dots, \vec{n}^{(k)})$ are defined.*

4.1. Katz-Liu computation. Katz and Liu in [5] computed the Open Gromov-Witten invariants for the Lagrangian of Large N duality associated to the unknot. In this particular case the Lagrangian is invariant by the S^1 -actions of the resolved conifold. The computation of Katz and Liu was based on the assumption that the virtual localization technique could be applied to Open Gromov-Witten invariants as in the closed case, without having a definition of Open Gromov-Witten invariant. It was realized that the result of the computation depends on the choice of the weight of the S^1 -action. Their result was in accord with the prediction of string theory after identifying the weight of the S^1 action with the frame of the knot.

Liu in [10] defines an invariant counting open curves associated to a triple (X, L, ρ) , where ρ is an admissible S^1 action on X (this means that ρ preserves L and acts freely on L). This invariant could be computed using the localization formula with respect to the S^1 -action ρ . In particular it gives a rigorous definition of the invariant computed in [5].

An admissible S^1 action defines a frame of the Lagrangian (in the sense of Definition 1): the associated element of $H_1(L')$ is the cycle given by an orbit of the S^1 -action. In the example of [5] we have then a one to one correspondence

$$(15) \quad \text{admissible } S^1\text{-actions} \leftrightarrow \text{frames of } L$$

The frame (15) is precisely the one for which the linking numbers of the orbits of the S^1 -action are zero.

Lemma 9. *The invariant of Liu [10] agrees with the Open Gromov-Witten invariants $F_{g, \vec{n}}(A)^f$ under the correspondence (15).*

The Lemma follows from the fact that the contribution of nontrivial (that is with at least an edge) decorated graphs is zero if we use of a perturbations invariants with respect to the inducted S^1 -action (as in [10]). We do not prove these statement, however it can be quickly understood if we assume that the localization technique can be applied to our case. The contribution of nontrivial graphs is computed in terms of linking numbers of boundaries components of curves. Since these components are orbits of the S^1 -action, for the choice of the frame (15) their linking number is zero as observed before.

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